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HYDRODYNAMICS WITH QUADRATIC PRESSURE. 1. GENERAL RESULTS

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A wide class of solutions of Euler equations with quadratic pressure are described. In Lagrangian coordinates, these solutions linearize exactly momentum equations and are characterized by special initial data: the Jacobian matrix of the initial velocity field has constant algebraic invariants. The equations are integrated using the method of separation of the time and Lagrangian coordinates. Time evolution is defined by elliptic functions. The solutions have a pole-type singularity at a finite time. A representation for the velocity vortex is given.

Introduction. For the Euler equations describing the motion of an ideal incompressible fluid with velocity field $\boldsymbol{u} = (u, v, w)$ and pressure p dependent on t and the space coordinates $\boldsymbol{x} = (x, y, z)$:

$$D\boldsymbol{u} + \nabla p = 0, \qquad \operatorname{div} \boldsymbol{u} = 0, \tag{1}$$

we seek solutions with pressure of the special form

$$p = k(t)r^2/2, \qquad r^2 = x^2 + y^2 + z^2. \tag{2}$$

The function k in (2) needs to be determined. We note that an arbitrary function of time can be added to the pressure in the form of (2).

Exact solutions in ideal fluid mechanics have been extensively studied. Andreev et al. [1] investigated the group properties of hydrodynamic models. Ovsyannikov [2] considered an exact solution of the Euler equations that was partly invariant with respect to the rotation group (so-called special vortex). We also cite papers [3–7], which give exact solutions of Euler equations.

Below we present several arguments in favor of studying exact solutions of hydrodynamic equations with pressure in the form of (2).

1. In studying barochronic motion of a gas $p = p(t)$, a consequence of momentum equations is the simple matrix Riccati equation $DJ + J^2 + kE = 0$ for the Jacobian matrix $J = \partial\boldsymbol{u}/\partial\boldsymbol{x}$, in which $k = 0$. Full information on the solution of this equation (eigenvalues and invariants, eigenvectors of the matrix J, and separation of the time variable and Lagrangian coordinates in the solution) provide for a complete description of the corresponding solutions in gas dynamics. Pressure in the form of (2) adds the scalar matrix kE to the Riccati equation. The scheme for studying barochronic solutions is applicable in this more complex situation, too. Separation of the time and Lagrangian variables in the solution remains a fundamental point. In this case, the rational functions of time are replaced by elliptic functions.

2. Cantwell [3], Popovich [6], and Abrashkin et al. [7] studied solutions of the Euler equations for pressure in the form of (2) and described them qualitatively. Cantwell [3] integrated the matrix equation for the elements of the Jacobian matrix, but did not integrate the Euler equations in finite form (with respect to the velocity components) and did not show that the solutions of the Lamé equations describe trajectories of fluid particles.

3. Classical objects of investigation in gas- and hydrodynamics are solutions with linear velocity fields, in which the pressure is a quadratic function of space variables [8, 9]. In this case, the gas- and hydrodynamic equations reduce to a system of ordinary differential equations. The examined solution with pressure in the form of (2) — a particular case of a quadratic function — does not reduce to solutions with linear velocity fields. The arbitrariness of this solution is three functions of two arguments.

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4. In recent studies in mathematical physics focus is on solutions that have singularities of the pole type [10]. This is due to the fact that the equations for elliptic functions and the Painlevé equation are the reduced equations (factor equations) in "large" models of mathematical physics. In hydrodynamic models, solutions of this type have not been studied in detail.

5. Meiermanov et al. [11] studied variants of the method of separation of time and Lagrangian variables. In [12], linear equations as factor equations were derived in studying partly invariant solutions of the Euler equations.

In the present paper, we give an algorithm for integrating the Euler equations and study general characteristics of their solution.

1. Relations for the Invariants. Let $J = \partial \boldsymbol{u}/\partial \boldsymbol{x}$ be the Jacobian matrix of the velocity field and k_i and λ_i $(i = 1, 2, 3)$ be algebraic invariants and eigenvalues of the matrix J, respectively. Then, the following relations hold:

$$k_1 = \sum_i \lambda_i = \mathrm{tr} J = 0, \quad k_2 = \sum_{i<j} \lambda_i \lambda_j = -\frac{1}{2} \sum_i \lambda_i^2 = -\frac{1}{2} \mathrm{tr} J^2,$$
$$k_3 = \lambda_1 \lambda_2 \lambda_3 = \frac{1}{3} \sum \lambda_i^3 = \frac{1}{3}\, \mathrm{tr} J^3. \tag{3}$$

The Hamilton–Cayley equation for the matrix J is written as

$$J^3 + k_2 J - k_3 E = 0, \tag{4}$$

where E is a unit matrix. A consequence of Eq. (1) with pressure in the form of (2) is the matrix equation

$$DJ + J^2 + kE = 0. \tag{5}$$

We can show that the sought-for motion of the fluid is defined by the properties of the Jacobian matrix J, which follow from the matrix Riccati equation (5).

Lemma 1. *The algebraic invariants k_2 and k_3 of the matrix J for the motion described are functions of time only and are defined from the system of ordinary differential equations*

$$k_2' - 3k_3 = 0, \quad k_3' + 2k_2^2/3 = 0 \tag{6}$$

reduced to the equation

$$k_2'^{\,2} = -4k_2^3/3 + C, \tag{7}$$

where C is an arbitrary constant.

The function $k = k(t)$ in (2) has the form

$$k = 2k_2/3. \tag{8}$$

Proof. Calculating the trace of the matrix from Eq. (5) with allowance for (3), we have $2k_2 - 3k = 0$ from which we obtain Eq. (8) and the dependence of k_2 only on t.

Multiplying Eq. (5) by the matrix J at the left and right and adding the relations obtained, we have the equation

$$DJ^2 + 2J^3 + 2kJ = 0. \tag{9}$$

Calculating the traces of both sides of Eq. (9), we obtain the first equation in (6), from which it follows that k_3 also depends only on t. Then, we multiply sequentially Eq. (5) by J^2 and J at the left and right. Adding the relations obtained, we have the equation for J^3:

$$DJ^3 + 3J^4 + 3kJ^2 = 0, \tag{10}$$

where J^4 can be expressed in terms of J and J^2 from Eq. (4) and from its consequence for J^4. Calculating the trace of Eq. (10) with allowance for (3) and the relation $\mathrm{tr} J^4 = 2k_2^2$, we obtain the second equation in (6).

Equation (7) can be obtained from system (6) by excluding the function k_3: by differentiating the first equation in (6) and substituting the values of k_3' from the second equation. The obtained equation $k_2'' = -2k_2^2$ is multiplied by k_2' and is integrated once to yield the equation for the elliptic Weierstrass function (7).

Remark 1. By extending the variables $t = a\tau$ and $k_2 = bq$, in which the constants a and b are determined from the equations

$$a^6 = 9\varepsilon C^{-1}, \qquad b = -3a^{-2}, \qquad \varepsilon = \pm 1, \tag{11}$$

using the homogeneity relations for elliptic functions [13], we reduce Eq. (7) to the form

$$\left(\frac{dq}{d\tau}\right)^2 = 4q^3 - 1. \tag{12}$$

The solution of Eq. (12) is an equianharmonic Weierstrass function [13] $q = \wp(\tau; 0, 1)$.

Remark 2. System (6) has an integral that relates the invariants k_2 and k_3:

$$k_3^2 + 4k_2^3/27 = C/9. \tag{13}$$

The constant C in the integral (13) [the same constant as in Eq. (7)] is proportional to the discriminant of the characteristic equation for the matrix J, $\lambda^3 + k_2\lambda - k_3 = 0$. By extending the variables (11), the integral (13) is reduced to

$$4q^3/27 + s^2 = \varepsilon/9, \tag{14}$$

where $s = ab^{-1}k_3$. Equation (14) is equivalent to (12).

Lemma 2. *Let $\lambda_i = \lambda_i(t)$ be eigenvalues of the Jacobian matrix J. Then, the following propositions are valid:*

(a) *System* (6) *is equivalent to the system for the eigenvalues*

$$\lambda_i' + \lambda_i^2 - \frac{1}{3}\sum_j \lambda_j^2 = 0, \qquad i = 1, 2, 3. \tag{15}$$

(b) *Let $q_i = q_i(t)$ be a logarithmic potential of the eigenvalue $\lambda_i = (\ln q_i)' = q_i'/q_i$. Then, q_i satisfies the Lamé equation*

$$q_i'' + k(t)q_i = 0, \tag{16}$$

where $k = 2k_2/3$ is an elliptic Weierstrass function.

Proof. (a) Using (3), we write system (6) in terms of λ_i. Then, solving this system with respect to λ_i', we obtain (15).

Conversely, (6) follows from (15). Multiplying each equation of (15) by λ_i and summing up the relations obtained, we have

$$\sum_i \lambda_i\lambda_i' + \sum_i \lambda_i^3 - \frac{1}{3}\sum_i \lambda_i\Big(\sum_j \lambda_j^2\Big) = \frac{1}{2}\Big(\sum \lambda_i^2\Big)' + \sum_i \lambda_i^3 = 0.$$

This is the first equation in (6). The second equation is obtained by multiplication of each equation of (15) by λ_i^2 and summation of the relations obtained. Indeed, using the relation $\sum_i \lambda_i^4 = \frac{1}{2}\Big(\sum \lambda_i^2\Big)^2$, we have

$$\sum_i \lambda_i^2\lambda_i' + \sum_i \lambda_i^4 - \frac{1}{3}\Big(\sum_j \lambda_j^2\Big)\Big(\sum_i \lambda_i\Big) = \frac{1}{3}\Big(\sum \lambda_i^3\Big)' + \frac{1}{6}\Big(\sum_i \lambda_i^2\Big)^2 = 0.$$

(b) Substituting the representations $\lambda_i = q_i'/q_i$ and $k = -\frac{1}{3}\sum_j \lambda_j^2$ into (15), we obtain Eq. (16).

Lemma 2 is proved.

Remark 3. By virtue of the relation $\sum \lambda_i = 0$, the logarithmic potentials are functionally dependent: $q_1q_2q_3 = \text{const}$. In addition, they are linearly dependent as three solutions of the second-order equation (16).

2. Representation of the Solution in Euler Coordinates. The results of Sec. 1 allow us to describe the solution completely.

Lemma 3. *Let q be a logarithmic potential of an eigenvalue of the matrix J* (*see proposition* "b" *of Lemma* 2). *Then, the vector*

$$\boldsymbol{\beta} = q'\boldsymbol{x} - q\boldsymbol{u} \tag{17}$$

is conserved along trajectories of fluid particles, i.e., depends only on Lagrangian coordinates.

Proof. We note that by virtue of the momentum equation in (1), for pressure in the form of (2), $D\boldsymbol{u} = -k\boldsymbol{x}$. Then,

$$D\boldsymbol{\beta} = q''\boldsymbol{x} - q'\boldsymbol{u} + q'\boldsymbol{u} - qD\boldsymbol{u} = (q'' + kq)\boldsymbol{x} = 0$$

by virtue of (16).

Remark 4. As follows from (17), the motion considered does not reduce to motion with a linear velocity field. Indeed, substituting the representation $\boldsymbol{u} = d\boldsymbol{x}/dt$ in Lagrangian coordinates into (17) and integrating the resulting relation, we obtain

$$\boldsymbol{x} = q[Q\boldsymbol{\beta}(\boldsymbol{x}_0) + \boldsymbol{\alpha}(\boldsymbol{x}_0)], \tag{18}$$

where $Q(t) = -\int q^{-2}(t)\,dt$, $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are vector-functions of the Lagrangian coordinates, determined from the initial conditions. Because the Lagrangian coordinates are determined with accuracy up to functional substitution, one of these vectors in the new coordinates can be reduced to $\boldsymbol{\beta}(\boldsymbol{x}_0) = \boldsymbol{x}_0'$. However, the second vector in (18) remains an arbitrary function of the new Lagrangian coordinates $\boldsymbol{\alpha} = \boldsymbol{\alpha}(\boldsymbol{x}_0')$, from which it follows that this solution does not reduce to a solution with a linear velocity field.

Lemma 4. *The initial field of velocities $\boldsymbol{u}_0(\boldsymbol{x}_0) = \boldsymbol{u}(0, \boldsymbol{x}_0)$ of the motion considered has a special form: the algebraic invariants of the Jacobian matrix $J_0 = \partial\boldsymbol{u}_0/\partial\boldsymbol{x}_0$ are real numbers.*

Vector fields of this type are described in the theory of barochronic motion of a gas [14].

We formulate the main result.

Theorem 1. *The velocity field for solution (2) of the Euler equations (1) is determined as an implicit vector-function from the system*

$$F_i(\boldsymbol{\beta}_i) = 0 \qquad (i = 1, 2, 3), \tag{19}$$

where $\boldsymbol{\beta}_i = q_i'\boldsymbol{x} - q_i\boldsymbol{u}$, $\lambda_i = q_i'/q_i$, and the functions q_i satisfy Eqs. (16).

The arbitrary functions F_i in (19) obey the condition of linear independence of the vectors $\nabla_{\boldsymbol{\beta}_i} F_i$. The arbitrariness of the solution obtained is three functions of two variables.

Proof. We show that the vector-function $\boldsymbol{u} = \boldsymbol{u}(t, \boldsymbol{x})$, which is a solution of (19), satisfies the momentum equations in (1) written for pressure in the form of (2) and, in addition, the matrix $J = \partial\boldsymbol{u}/\partial\boldsymbol{x}$ has algebraic invariants described by Lemma 1.

We apply the operation D to Eq. (19). Then, as $D\boldsymbol{\beta}_i = q_i''\boldsymbol{x} + q_i'\boldsymbol{u} - q_i'\boldsymbol{u} - q_iD\boldsymbol{u} = q_i''\boldsymbol{x} - q_iD\boldsymbol{u}$, by virtue of the linear independence of vectors $\nabla_{\boldsymbol{\beta}_i} F_i$, we have $D\boldsymbol{\beta}_i = 0$ from the equation $D\boldsymbol{\beta}_i\nabla_{\boldsymbol{\beta}_i} F_i = 0$. From this, we obtain dependence (2) for pressure and, consequently, all the results considered in Sec. 1.

Let $T = (\nabla_{\boldsymbol{\beta}_i} F_i)$ be a matrix whose rows are the components of the corresponding gradients of the function F_i. Differentiating Eq. (19) with respect to all space variables, we obtain the matrix relation

$$T\,\frac{\partial\boldsymbol{\beta}_i}{\partial\boldsymbol{x}} = 0,$$

from which, by virtue of $\partial\boldsymbol{\beta}_i/\partial\boldsymbol{x} = q_i'E - q_iJ$, it follows that

$$TJ = \Lambda T, \tag{20}$$

where $\Lambda = \operatorname{diag}(\lambda_i)$ is the Jordan form of the matrix J. By virtue of (20), the matrices J and Λ are similar, and, hence, their algebraic invariants coincide. In this case, T is a transforming matrix, and the vectors of the row $\nabla_{\boldsymbol{\beta}_i} F_i$ are the left eigenvectors of the matrix J, which correspond to the eigenvalues λ_i.

Remark 5. If the matrix J has a pair of complex-conjugate eigenvalues, Eqs. (19) relate complex-valued functions. The theory of barochronic motion of a gas shows how to obtain the real form of the solution in this case [14].

Remark 6. Since the relation $\sum \lambda_i = 0$ is satisfied for the matrix J, the eigenvalue of multiplicity of 3 is impossible. For an eigenvalue of multiplicity of 2, there is a simple representation for λ_i. Let $\lambda_1 = \lambda_2 = \lambda$ and $\lambda_3 = -2\lambda$. Then, (15) reduces to one equation, whose solution is $\lambda = \lambda_0/(1 + \lambda_0(t_0 - t))$, where the constant $\lambda_0 = \lambda(t_0)$ is the initial value of λ. In this case, $\operatorname{rank}(J - \lambda E) = 1$, and, hence, the multiple eigenvalue corresponds to a pair of linearly independent left eigenvectors, in terms of which solution (19) is expressed.

Thus, the algorithm for solving the Euler equations (1) with pressure in the form of (2) includes the following steps.

Step 1. An algebraic structure of the Jacobian matrix J_0 for the initial velocity field is specified: the real numbers k_{20} and k_{30} (or λ_{i0} are such that $\sum \lambda_{i0} = 0$).

Step 2. The initial velocity field $\boldsymbol{u}_0 = \boldsymbol{u}_0(\boldsymbol{x}_0)$ is constructed as an implicit function which is a solution of the equation $F_i(\boldsymbol{u}_0 - \lambda_{i0}\boldsymbol{x}_0) = 0$, where the functions F_i satisfy the conditions of Theorem 1.

Step 3. Equation (16) [or system (6)] is integrated subject to the specified initial data (see Step 1).

Step 4. The velocity field $\boldsymbol{u} = \boldsymbol{u}(t, \boldsymbol{x})$ is determined as a solution of (19).

Therefore, to find solutions of the form (2), we need to have a sufficiently large number of initial vector fields (see Step 2) and know how to integrate Eq. (16) or Eq. (6). The Lamé equation in the form of (16) is used to describe the sought-for motion.

Remark 7. This solution of the Euler equations is of group-theoretical origin and can be obtained using a two-step algorithm. In the first step, we find a solution of the Euler equations of the "special vortex" type [2]. This is a partly invariant solution with respect to the rotation group of rank 2 and defect 1. In the second step, we seek an invariant solution of the obtained factor-system for the assumed dilation operator $r\partial_r + U\partial_U + 2p\partial_p$ (U is the radial velocity component). The invariant representation for pressure has the form of (2). Although this solution can be analyzed using the formulas describing the "special vortex" [2], this is difficult because of the complex expression for the angular component ω, which is a tangent to the spheres of the velocity components.

3. Description of the Solution in Lagrangian Coordinates. In Lagrangian coordinates, Eqs. (1) for the sought-for function $\boldsymbol{x} = \boldsymbol{x}(t, \boldsymbol{x}_0)$ with pressure in the form of (2) take the form

$$\frac{d^2\boldsymbol{x}}{dt^2} + k(t)\boldsymbol{x} = 0 \qquad (21)$$

and describe trajectories of fluid particles subject to the initial data $\boldsymbol{x}(0) = \boldsymbol{x}_0$ (Lagrangian coordinates).

Theorem 2. *The motion of a fluid described by the solution at hand has the following properties:*

(a) *The motion of fluid particles is described by the Lamé equation* (21).

(b) *The motion of each fluid particle occurs in a plane* Π *whose position in the space* $\mathbb{R}^3(\boldsymbol{x})$ *is determined by the initial data for this particle:*

$$\Pi\colon \boldsymbol{x} \times \frac{d\boldsymbol{x}}{dt} = \boldsymbol{B}_0, \quad \boldsymbol{B}_0 = \boldsymbol{x}_0 \times \boldsymbol{u}_0, \qquad (22)$$

where "×" is the sign of the vector product and $\boldsymbol{u}_0$ *is the initial velocity of a fluid particle that at* $t = 0$ *has position* $\boldsymbol{x}_0$.

Proof. The item (a) is proved above. It is easy to verify that $\boldsymbol{B}_0$ specifies the integral of the system

$$\frac{d}{dt}\boldsymbol{B}_0 = \frac{d}{dt}\Big(\boldsymbol{x} \times \frac{d\boldsymbol{x}}{dt}\Big) = \boldsymbol{x} \times \frac{d^2\boldsymbol{x}}{dt^2} = -k\boldsymbol{x} \times \boldsymbol{x} = 0$$

by virtue of Eq. (21).

There is an analogy between the motion of a fluid particle and the motion of a material point under the action of the central force $\boldsymbol{F} = -k\boldsymbol{x}$ [15].

Lemma 5. *After conversion to the polar coordinates* $x = R\cos\theta$ *and* $y = R\sin\theta$, *the equations of motion of a fluid particle in the plane* Π: $z = 0$, $xy' - yx' = l_0$, *where* l_0 *is a function of the Lagrangian coordinates, take the form*

$$\frac{d\theta}{dt} = \frac{l_0}{R^2}, \qquad \frac{d^2R}{dt^2} + k(t)R = \frac{l_0^2}{R^3}. \qquad (23)$$

The second equation in (23) *is Ermakov's equation.*

Proof. By rotation in the space $\mathbb{R}^3(\boldsymbol{x})$, which is individual for each particle and depends on the vector $\boldsymbol{B}_0$, the plane of motion of the fluid particle can be reduced to the plane Π ($z = 0$). Then, integral (22) takes the form given in Lemma 5, and l_0 depends only on the initial data (Lagrangian coordinates). Conversion to polar coordinates is implemented in a standard manner. Recently, the relationship between the equation of harmonic oscillator (21) and Ermakov's equation (23) has been discussed in many papers (see, for example [16]). The second equation in (23) describes the evolution of the radius-vector of a fluid particle in spherical coordinates, too. This means that conversion to this system does not facilitate analysis of the solutions considered.

Lemma 6. *The eigenvectors of the matrix* J (*both left* $\boldsymbol{l}_i$ *and right* $\boldsymbol{r}_i$) *can be constant along trajectories of fluid particles, i.e., dependent on Lagrangian coordinates.*

Proof. Let $\boldsymbol{l}$ be the left eigenvector of the matrix J that corresponds to the eigenvalue λ: $\boldsymbol{l}J = \lambda\boldsymbol{l}$. We apply the operator D to this equality:

$$D\boldsymbol{l}J + \boldsymbol{l}DJ = \lambda'\boldsymbol{l} + \lambda D\boldsymbol{l}.$$

Substituting DJ from (5) and $\lambda' = -\lambda^2 - k$ from (15) into this equation and using $\boldsymbol{l}J^2 = \lambda^2\boldsymbol{l}$, we obtain $D\boldsymbol{l}J = \lambda D\boldsymbol{l}$. Because the invariant space corresponding to the vector $\boldsymbol{l}$ is one-dimensional, it follows that $D\boldsymbol{l} = b\boldsymbol{l}$, where $b = b(t, \boldsymbol{x}) \neq 0$. This equation can be integrated in the form $\boldsymbol{l} = K(t, \boldsymbol{x})\boldsymbol{l}_0$, where $D\boldsymbol{l}_0 = 0$. Because the proper vector is determined with accuracy up to a factor, we can choose $\boldsymbol{l}$ such that it depends only on Lagrangian coordinates. The proof for the right eigenvector $\boldsymbol{r}$ is similar.

Lemma 7. *In the motion studied, the vortex* $\boldsymbol{\omega} = \operatorname{rot}\boldsymbol{u}$ *is written as*

$$\boldsymbol{\omega} = \sum_{i=1}^{3} \varepsilon_i q_i \boldsymbol{r}_i, \tag{24}$$

where q_i *is the logarithmic potential of the eigenvalue* λ_i *and* $\varepsilon_i = 0$ *and* 1.

Proof. The equation for the vortex $\boldsymbol{\omega} = \boldsymbol{\omega}(t, \boldsymbol{x})$ can be written as

$$D\boldsymbol{\omega} - J\boldsymbol{\omega} = 0.$$

From representation (23), $D\boldsymbol{\omega} = \sum_i \varepsilon_i q_i' \boldsymbol{r}_i$ and $\boldsymbol{\omega} = \sum \varepsilon_i q_i J\boldsymbol{r}_i = \sum \varepsilon_i q_i \lambda_i \boldsymbol{r}_i$. The factors ε_i appear in the formula for the vortex because $\boldsymbol{r}_i \neq 0$ and $q_i \neq 0$ by definition. Lemma 6 is proved.

4. General Solution of the Lamé Equation. After substitution of variables (11), Eq. (16) [or each equation in (21)] takes the form

$$\frac{d^2 q}{d\tau^2} - 2\wp(\tau) q = 0, \tag{25}$$

where $\wp(\tau) = \wp(\tau; 0, 1)$ is a solution of Eq. (12). Necessary information on the properties of solutions of the Lamé equations is given in [17, 18], and that on the elliptic functions is given in [13].

The general integral of Eq. (25) is a meromorphic function. According to the Picard theorem, this equation is integrated using second-order, doubly periodic functions with the same periods 2ω and $2\omega'$ as in elliptic functions that are the coefficients of the equation. Thus, for the general integral of Eq. (25), $q = Q(\tau)$, the formulas

$$Q(\tau + 2\omega) = \mu Q(\tau), \qquad Q(\tau + 2\omega') = \mu' Q(\tau)$$

are valid. Here the constants μ and μ' are the so-called factors of the function Q.

Let z_0 be a root of the equation $\wp(z_0) = 0$. Then, Eq. (25) has a fundamental system of solutions

$$q_j(\tau) = \mathrm{e}^{\mp\tau\zeta(z_0)} \sigma(\tau \pm z_0)/\sigma(\tau), \qquad j = 1, 2. \tag{26}$$

The elliptic Weierstrass function $\wp = \wp(\tau; 0, 1)$ is defined on a complex plane and is a one-valued, doubly periodic, analytical function which has a double pole at the point $\tau = 0$. In this equianharmonic case, the parallelogram of the periods on the complex plane has the vertices 0, 2ω, $2\omega_2$, and $2\omega'$ (anticlockwise orientation), where $\omega_2 = \omega + \omega' \in \mathbb{R}$, 2ω and $2\omega'$ are the periods, and $\omega' = \omega^*$. In this case, $\omega_2 \approx 1.5299$ is a real number, $2\omega = \omega_2 + i\omega_2\sqrt{3}$, $z_0 = \omega_2 + i\omega_2/\sqrt{3}$, and $\zeta(z_0) = ((\pi/3)\omega_2)\exp(-i\pi/6)$. On the real axis $\tau \in \mathbb{R}$, the elliptic function $\wp = \wp(\tau; 0, 1)$ also takes real values.

Let $\boldsymbol{x}_0$ and $\boldsymbol{u}_0 = \boldsymbol{u}_0(\boldsymbol{x}_0)$ be the initial data for a fluid particle at $\tau = \tau_0$ and $W = q_1 q_2' - q_2 q_1'$ be the Wronskian of the fundamental system of solutions (26), and $W(\tau) = W(\tau_0) = W_0$ is a nonzero constant. Then, the general solution of system (21) is written as

$$\boldsymbol{x} = W_0^{-1}[(q_{20}' q_1 - q_{10}' q_2)\boldsymbol{x}_0 + (q_{10} q_2 - q_{20} q_1)\boldsymbol{u}_0(\boldsymbol{x}_0)], \tag{27}$$

where the subscript 0 denotes the values of the corresponding functions at $\tau = \tau_0$.

Investigation of the properties of the solution in the form of (27) is an independent problem. We can conclude *a priori* that the general solution has some singularities of the type of the poles of the elliptic function $\wp(\tau; 0, 1)$. In this solution, the isobars are spheres whose radii depend on time. Generally, the formulas of solution (27) do not inherit the spherical symmetry of the pressure distribution. The initial distribution of the velocity field influences significantly the evolution of the solution. Equation (24) for the velocity vortex gives valuable information on the evolution of the solution. This equation defines the direction along which the vortex grows without bound.

Conclusions. 1. A complete analytical description is given for a broad class of exact solutions of the Euler equations describing the motion of an ideal incompressible fluid for which the pressure is proportional to the squared distance.

2. It is shown that in Euler coordinates, the motion is described by finite formulas that define the velocity field as an implicit vector-function. The motion dynamics is specified by elliptic functions of time. The properties of the motion are largely determined by the Jacobian matrix of the velocity field of special form: its algebraic invariants depend only on time. The Jacobian matrix of the initial velocity field has constant algebraic invariants. The eigenvectors of the Jacobian matrix are constant along the trajectories.

3. The Euler equations are integrated using a variant of the method of separation of variables. Elliptic functions define the dependence of the desired parameters (velocity field, pressure, and vortex) on time, Lagrangian coordinates specify the initial distribution and velocity of fluid particles, and the initial velocity field is special.

4. The equations of trajectories of fluid particles in Lagrangian coordinates are reduced to the Lamé equations, which are integrated in second-order, doubly periodic functions. The vortex vector is represented as a linear combination of the eigenvectors of the Jacobian matrix. A special feature of the motion is that it does not inherit the symmetry of the spherically symmetric pressure distribution.

5. For the solutions derived, the factor equations of the Euler equations are linear second-order equations. The method of separation of variables applied to these equations yields the formula for the general solution.

6. The solutions have singular points t_* that correspond to the poles of the elliptic functions. Thus, in the vicinity of the corresponding times, the pressure and the vector components tend to infinity. Physical interpretation of the motion of the fluid is obviously possible only for times $t > t_*$ or $t < t_*$.

7. If the Jacobian matrix $J = \partial \boldsymbol{u}/\partial \boldsymbol{x}$ has a multiple eigenvalue, the equations of trajectories are integrated over time in elementary functions.

In the forthcoming paper, we will give examples illustrating the behavior of trajectories of fluid particles for various initial data and the evolution of an elementary spherical fluid volume.

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